MCR 3U Final Exam Prep

1. Unit 1 - Quadratics

Factoring:

- always look for a common factor FIRST and take it out (by division)
- look for patterns

a) common factor (no 'c')

$3x^2 - 18x$
$= 3x(x-6)$

$-5x^2 + 15x$
$= -5x(x-3)$

b) difference of squares (no 'b')

square roots

$25x^2 - 49$
$= (5x-7)(5x+7)$

$4x^2 - 49$
$= (2x-7)(2x+7)$

c) simple trinomial x^2+bx+c

$\_ \times \_ = c$
$\_ + \_ = b$

$x^2 - 6x - 27$
$= (x-9)(x+3)$

$2x^2 - 2x - 24$
$= 2(x^2-x-12)$
$= 2(x-4)(x+3)$

d) trinomial square

square roots

$4x^2 + 20x + 25$
$= (2x+5)^2$

$49x^2 - 42x + 9$
$= (7x-3)^2$

e) complex trinomial ax^2+bx+c

charts or decomposition

$2x^2 - 7x - 15$ $\quad -10 \times 3 = -30$, $\; -10 + 3 = -7$
$= 2x^2 - 10x + 3x - 15$
$= 2x(x-5) + 3(x-5)$
$= (2x+3)(x-5)$

OR

	x	-5
$2x$	$2x^2$	$-10x$
$+3$	$+3x$	-15

$10x^2 - 28x + 16$ $\quad -10 \times -4 = 40$, $\; -10 + -4 = -14$
$= 2(5x^2 - 14x + 8)$
$= 2(5x^2 - 10x - 4x + 8)$
$= 2(5x(x-2) - 4(x-2))$
$= 2(5x-4)(x-2)$

OR

	x	-2
$5x$	$5x^2$	$-10x$
-4	$-4x$	$+8$

Problem Solving:

Types of Problems	Key Words to Look For	Mathematical Strategy
1. Optimization	-maximum, minimum, most, least, highest, lowest, etc.	-CTS or shortcut (-b/2a)
2. Solving	-how long, how far, how many, when, break even points, etc.	-SF: quadratic formula -VF: isolate ind. var.
3. Determining if something could happen	-is it possible, will this happen, could this happen, etc.	-find optimal value and compare OR analyze discriminant

A water balloon is catapulted into the air so that its height, h, in metres, after t seconds is $h(t) = -4.9t^2 + 25t + 1.9$.

a) For how long is the balloon above 13 m?

$13 = -4.9t^2 + 25t + 1.9$

$0 = -4.9t^2 + 25t - 11.1$

$t = \dfrac{-25 \pm \sqrt{(25)^2 - 4(-4.9)(-11.1)}}{2(-4.9)}$

$t = \dfrac{-25 \pm \sqrt{407.44}}{-9.8}$

$t = \dfrac{-25 + \sqrt{407.44}}{-9.8}$ $\quad$ $t = \dfrac{-25 - \sqrt{407.44}}{-9.8}$

$t = 0.49$ s $\quad$ $t = 4.61$ s

∴ above 13m for

$4.61 - 0.49$

$= 4.12$ seconds

b) Is it possible for the balloon to reach a height of 15 m?

Find max height + compare

$t = \dfrac{-25}{2(-4.9)}$

$t = 2.55$

$h(2.55) = -4.9(2.55)^2 + 25(2.55) + 1.9$

$= 33.79$ m

this is greater than 15m

∴ yes it is possible

Or substitute and use discriminant

$15 = -4.9t^2 + 25t + 1.9$

$0 = -4.9t^2 + 25t - 13.1$

$b^2 - 4ac = (25)^2 - 4(-4.9)(-13.1)$

$= 368.24$

positive

∴ yes it is possible to reach 15m

Inequalities:

- same rules as solving equations
- EXCEPT for when you multiply or divide by a negative - change direction

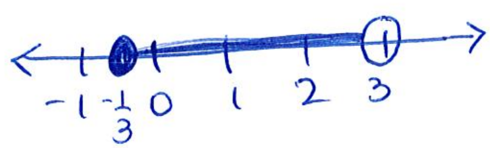
Solve: $-4 < \frac{1-3x}{2} \leq 1$

$$-8 < 1-3x \leq 2$$

$$\frac{-9}{-3} < \frac{-3x}{-3} \leq \frac{1}{-3}$$

$$3 > x \geq -\frac{1}{3}$$

OR $-\frac{1}{3} \leq x < 3$



Radicals:

- simplify to get mixed radicals
- like radicals can be combined by addition or subtraction

Simplify $3\sqrt{27} + 4\sqrt{75} - 2\sqrt{32}$

$$= 3\sqrt{9 \times 3} + 4\sqrt{25 \times 3} - 2\sqrt{16 \times 2}$$

$$= 3(3\sqrt{3}) + 4(5\sqrt{3}) - 2(4\sqrt{2})$$

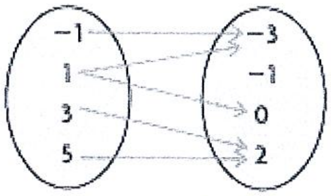
$$= 9\sqrt{3} + 20\sqrt{3} - 8\sqrt{2}$$

$$= 29\sqrt{3} - 8\sqrt{2}$$

2. Unit 2 - Functions, Function Notation, Transformations, D&R

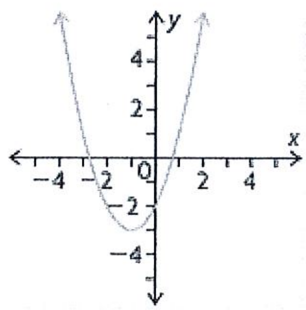
Functions:

- every x has one distinct y



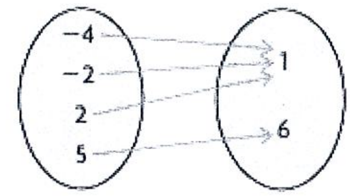
x value 1 has 2 different y values

∴ not a function



passes VLT

∴ function



no duplication of x's

∴ function

Function Notation:

- Substitute and evalute

Evaluate and simplify for $f(x) = 2x^2 - 3x + 1$

a) $f(2) = 2(2)^2 - 3(2) + 1$

$= 8 - 6 + 1$

$= 3$

b) $3f(-1) = 3\left[2(-1)^2 - 3(-1) + 1\right]$

$= 3(2 + 3 + 1)$

$= 3(6)$

$= 18$

c) $f(x-4) = 2(x-4)^2 - 3(x-4) + 1$

$= 2(x^2 - 8x + 16) - 3x + 12 + 1$

$= 2x^2 - 16x + 32 - 3x + 13$

$= 2x^2 - 19x + 45$

D & R/Transformations

- Use set notation for D & R
- Don't forget to factor where necessary

a) $y = -4\sqrt{0.5(x+5)} - 1$

- refl. in x axis
- v stretch of 4
- h stretch of 2
- left 5
- down 1

D: $\{x \mid x \geq -5, x \in R\}$

R: $\{y \mid y \leq -1, y \in R\}$

b) $y = \frac{0.5}{3(x-2)} + 1$

- v comp of 1/2
- h comp of 1/3
- right 2
- up 1

D: $\{x \mid x \neq 2, x \in R\}$

R: $\{y \mid y \neq 1, y \in R\}$

c) $y = -6|-0.1(x-1)| + 9$

- refl in x axis
- v stretch of 6
- refl in y axis
- h stretch of 10
- right 1
- up 9

D: $\{x \mid x \in R\}$

R: $\{y \mid y \leq 9, y \in R\}$

d) $y = \frac{1}{3}\sqrt{-2x+14} - 10$

$-2(x-7)$

- v comp of 1/3
- refl. in y axis
- h comp of 1/2
- right 7
- down 10

D: $\{x \mid x \leq 7, x \in R\}$

R: $\{y \mid y \geq -10, y \in R\}$

3. Unit 3 - Inverse and Rational Expressions

Inverse Functions:

- Switch variables and isolate y (use opposite operations)
- In context, DO NOT switch variables - just isolate independent variable

1. For $f(x) = \frac{2}{3}(x - 5)$, determine $f^{-1}(-2)$

$y = \frac{2}{3}(x-5)$

$x = \frac{2}{3}(y-5)$

$\frac{3}{2}x = y-5$

$\frac{3}{2}x + 5 = y$

$\therefore f^{-1}(x) = \frac{3}{2}x + 5$

$f^{-1}(-2) = \frac{3}{2}(-2) + 5$

$= -3 + 5$

$= 2$

2. Determine the inverse of $f(x) = \frac{1}{2}(x - 5)^2 + 3$

$y = \frac{1}{2}(x-5)^2 + 3$

$x = \frac{1}{2}(y-5)^2 + 3$

$x - 3 = \frac{1}{2}(y-5)^2$

$2(x-3) = (y-5)^2$

$\pm\sqrt{2(x-3)} = y - 5$

$5 \pm \sqrt{2(x-3)} = y$

$\therefore f^{-1}(x) = 5 \pm \sqrt{2(x-3)}$

*don't forget the ± when you sq. root

3. The height of a rocket launched from the ground can be modelled by the function $h(t) = -5(t - 6)^2 + 65$, where h(t) is the height above the ground, in metres, and t is the time in seconds after it is launched. Determine the model that describes time in terms of height.

$h = -5(t-6)^2 + 65$

$\frac{h-65}{-5} = \frac{-5(t-6)^2}{-5}$

$\frac{h-65}{-5} = (t-6)^2$

$\pm\sqrt{\frac{h-65}{-5}} = t - 6$

$6 \pm \sqrt{\frac{h-65}{-5}} = t$

*make sure you show the square root over the entire fraction

Rational Expressions:

- Factor <u>all</u> parts
- Cancel where possible
- State simplified expression
- State restrictions

Simplify and state any restrictions

a) $\frac{3x+9}{-5x-15}$

$= \frac{3\cancel{(x+3)}}{-5\cancel{(x+3)}}$

$= -\frac{3}{5}, x \neq -3$

b) $\frac{16x^2-9}{4x^2+x-3}$

$= \frac{\cancel{(4x-3)}(4x+3)}{\cancel{(4x-3)}(x+1)}$

$= \frac{4x+3}{x+1}, x \neq \frac{3}{4}, -1$

c) $\frac{3}{x^2-9} \div \frac{3x-6}{x-3}$

$= \frac{\cancel{3}}{\cancel{(x-3)}(x+3)} \times \frac{\cancel{(x-3)}}{\cancel{3}(x-2)}$

$= \frac{1}{(x+3)(x-2)}, x \neq \pm 3, 2$

d) $\frac{x^2+4x-21}{x^2-6x-16} \times \frac{x^2-8x+15}{x^2+9x+14}$

$= \frac{\cancel{(x+7)}(x-3)}{(x-8)(x+2)} \times \frac{(x-3)(x-5)}{\cancel{(x+7)}(x+2)}$

$= \frac{(x-3)^2(x-5)}{(x+2)^2(x-8)}, x \neq 8, -2, -7$

e) $\frac{3x^2-7x-6}{6x^2+3x} \times \frac{9x-6}{2x^2-5x-3} \div \frac{9x^2-4}{4x^2+4x+1}$

$= \frac{\cancel{(3x+2)}\cancel{(x-3)}}{\cancel{3}x\cancel{(2x+1)}} \times \frac{\cancel{3}\cancel{(3x-2)}}{\cancel{(2x+1)}\cancel{(x-3)}} \times \frac{\cancel{(2x+1)}\cancel{(2x+1)}}{\cancel{(3x-2)}\cancel{(3x+2)}}$

$= \frac{1}{x}, x \neq 0, -\frac{1}{2}, 3, \pm\frac{2}{3}$

Discontinuities:

- If it cancels, there is a hole at the location of the restriction
- If it is still in the denominator, there is a VA at the location of the restriction

State the discontinuities for part b above

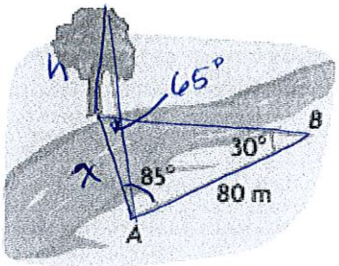
hole @ $x = \frac{3}{4}$

VA @ $x = -1$

4. Unit 4 – Trigonometry

Problem Solving:

- Sine law/cosine law/SOH CAH TOA
- For sine law problems with SSA information, always check for Ambiguous Case



In order to find the height of a tree across the river, Bert lays out a baseline 80 m long and measures the angles as shown in the diagram. The angle of elevation from A to the top of the tree is 28°. Determine the height of the tree, to one decimal place.

$$\frac{x}{\sin 30} = \frac{80}{\sin 65}$$

$$x = \frac{80\sin 30}{\sin 65}$$

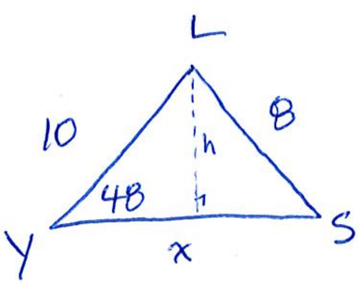
$$x = 44.1\text{ m}$$

$$\tan 28 = \frac{h}{44.1}$$

$$h = 44.1\tan 28$$

$h = 23.4\text{m}$ ← height of tree

A lighthouse at point L is 10 km from a yacht at point Y and 8 km from a sailboat at point S. From the yacht, the lighthouse and the sailboat are separated by an angle of 48°. Determine the distance from the yacht to the sailboat, to the nearest tenth of a kilometre.



SSA ∴ check height to determine # of solutions

$h = 10\sin 48$
$h = 7.4$

since $h < 8 < 10$
there are two possible solutions

CASE 1

$$\frac{\sin S}{10} = \frac{\sin 48}{8}$$

$$S = \sin^{-1}\left(\frac{10\sin 48}{8}\right)$$

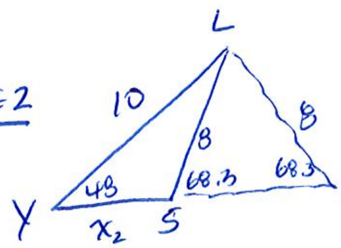
$S = 68.3°$ ∴ $L = 63.7°$

$$\frac{x}{\sin 63.7} = \frac{8}{\sin 48}$$

$$x = \frac{8\sin 63.7}{\sin 48}$$

$$x = 9.7\text{ km}$$

CASE 2



$S = 180 - 68.3$
$S = 111.7°$
∴ $L = 20.3°$

$$\frac{x_2}{\sin 20.3} = \frac{8}{\sin 48}$$

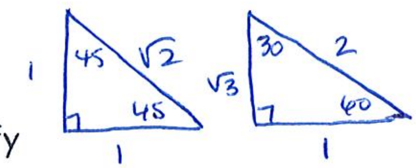
$$x_2 = \frac{8\sin 20.3}{\sin 48}$$

$$x_2 = 3.7\text{ km}$$

∴ the sailboat could have been 9.7km or 3.7km from the yacht

Special Angles:

- 30, 45, and 60 degree angles
- Use exact values and show your steps to simplify



Simplify each of the following

a) sin 30 cos 30 tan 30 - 2cos 45

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{3}\right) - 2\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{3}{12} - \sqrt{2}$

$= \frac{1}{4} - \frac{4\sqrt{2}}{4}$

$= \frac{1-4\sqrt{2}}{4}$

b) $4\cos^2 60 + 6\tan 30$

$= 4\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{3}}{3}\right)$

$= 4\left(\frac{1}{4}\right) + 2\sqrt{3}$

$= 1 + 2\sqrt{3}$

Solving Angles:

- Related acute angle must be positive and less than 90°
- Use CAST and Quadrant rules to find all solutions in the given domain

Solve for x, given $0° \leq x \leq 360°$

a) 5 - 3sinx = 4

$\frac{-3\sin x}{-3} = \frac{-1}{-3}$

$\sin x = \frac{1}{3}$ ← positive

S | A
T | C

$x = \sin^{-1}\left|\frac{1}{3}\right|$

$x = 19.5°$ ← RAA

Quad I : $x_1 = 19.5°$

Quad II : $x_2 = 180 - 19.5$

$x_2 = 160.5°$

b) 7 cosx + 3 = 2 cosx + 1

$7\cos x - 2\cos x = 1 - 3$

$\frac{5\cos x}{5} = \frac{-2}{5}$

$\cos x = -\frac{2}{5}$ ← negative

S | A
T | C

$x = \cos^{-1}\left|\frac{-2}{5}\right|$

$x = 66.4°$ ← RAA

Quad II : $x_1 = 180 - 66.4$

$x_1 = 113.6°$

Quad III : $x_2 = 180 + 66.4$

$x_2 = 246.4°$

Identities:

- Always separate LS and RS
- Use a combination of good math manipulations and substituting known identities

Prove the following

a) $\frac{1}{\tan x} + \tan x = \frac{1}{\sin x \cos x}$

$LS = \cot x + \tan x$ $\quad RS = \frac{1}{\sin x \cos x}$

$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$

$= \frac{1}{\sin x \cos x}$

$LS = RS$

b) $\sin x - \sin x \cos^2 x = \sin^3 x$

$LS = \sin x - \sin x \cos^2 x$ $\quad RS = \sin^3 x$

$= \sin x (1 - \cos^2 x)$

$= \sin x (\sin^2 x)$

$= \sin^3 x$

$LS = RS$

c) $\frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} = 2\tan x$

$LS = \frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}$ $\quad RS = 2\tan x$

$= 2\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x(1+\sin x) - \cos x(1-\sin x)}{(1-\sin x)(1+\sin x)}$

$= \frac{\cos x(1+\sin x - (1-\sin x))}{\cos^2 x}$

$= \frac{1+\sin x - 1 + \sin x}{\cos x}$

$= \frac{2\sin x}{\cos x}$

$LS = RS$

d) $\frac{\tan^2 x}{\tan^2 x+1} = \sin^2 x$

$LS = \frac{\tan^2 x}{\tan^2 x + 1}$ $\quad RS = \sin^2 x$

$= \frac{\sin^2 x}{\cos^2 x} \div \frac{\sin^2 x}{\cos^2 x} + 1$

$= \frac{\sin^2 x}{\cos^2 x} \div \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$

$= \frac{\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{1}$

$= \sin^2 x$

$LS = RS$

5. Unit 5 – Periodic Functions

- factor where necessary

Function	Amplitude	Period (in degrees)	Max	Min	Phase Shift	Vertical Shift
$\frac{1}{4}(\theta-120)$ $y=-3sin\left(\frac{1}{4}\theta-30\right)+3$	3	1440	6	0	right 120	up 3
$y=0.25cos(2(\theta+90))-1.25$	0.25	180	-1	-1.5	left 90	down 1.25
$5(\theta-50)$ $y=-\frac{1}{2}sin(5\theta-250))+2.5$	$\frac{1}{2}$	72	3	2	right 50	up 2.5

6. Unit 6 – Exponential Functions

Power Rules:

- Usually power of a power first, but depends on specific question
- Only evaluate at the end
- Know how to convert radicals to exponents

Simplify then evaluate:

$$\frac{(4^{-5})^3(4)}{(4^8)^{-2}}$$

$$= \frac{4^{-15}4^{1}}{4^{-16}}$$

$$= \frac{4^{-14}}{4^{-16}}$$

$$= 4^2$$

$$= 16$$

$$\left(\frac{8^{17}}{8^{19}}\right)^2\left(\frac{8^{-5}}{8^{-6}}\right)^3$$

$$= (8^{-2})^2(8^1)^3$$

$$= (8^{-4})(8^3)$$

$$= 8^{-1}$$

$$= \frac{1}{8}$$

Simplify, expressing your solution as an exact value (no decimals) with no negative exponents:

$$\sqrt[5]{\frac{(3x^4)^{-2}(3x)^9}{(3x^{-2})^2}}$$

$$= \left(\frac{3^{-2}x^{-8}\,3^9x^9}{3^2x^{-4}}\right)^{1/5}$$

$$= \left(\frac{3^7x^1}{3^2x^{-4}}\right)^{1/5}$$

$$= \left(3^5x^5\right)^{1/5}$$

$$= 3^1x^1$$

$$= 3x$$

$$\frac{21x^{-3}y^8}{(3x^3y^{-2})^2}$$

$$= \frac{21x^{-3}y^8}{3^2x^6y^{-4}}$$

$$= \frac{21y^{12}}{9x^9}$$

$$= \frac{7y^{12}}{3x^9}$$

Solving Exponential Equations:

- Rewrite so the bases are the same, then equate the exponents to solve

Solve:

$4^{3x+1} = 16^{x-1}$

$$4^{3x+1} = (4^2)^{x-1}$$

$$4^{3x+1} = 4^{2x-2}$$

$$\therefore 3x+1 = 2x-2$$

$$3x-2x = -2-1$$

$$x = -3$$

$\frac{1}{125} = (5^{x+4})^x$

$$5^{-3} = 5^{x^2+4x}$$

$$\therefore -3 = x^2+4x$$

$$0 = x^2+4x+3$$

$$0 = (x+3)(x+1)$$

$$\therefore x = -3$$

$$x = -1$$

7. Unit 7 - Sequences and Series

Find t_n, t_{10}, and S_{11} for the sequence 16, 35, 54, 73, ...

$a = 16$
$d = 19$

$t_n = 16 + (n-1)(19)$
$= 16 + 19n - 19$
$t_n = -3 + 19n$

$S_{11} = \frac{11}{2}(2(16) + (11-1)(19))$
$= 1221$

$t_{10} = -3 + 19(10)$
$= 187$

Find t_n and t_8 for the sequence -6, 12, -24, ...

$a = -6$
$r = -2$

$t_n = -6(-2)^{n-1}$

$t_8 = -6(-2)^{8-1}$
$= 768$

Find the sum of the geometric series 49 152 + 12 288 + ... + 12

$a = 49152$
$r = 0.25$
$t_n = 12$

$\frac{12}{49152} = \frac{49152(0.25)^{n-1}}{49152}$

$\frac{\log \frac{12}{49152}}{\log 0.25} + 1 = n$

$6 + 1 = n$
$7 = n$

$S_7 = \frac{49152(0.25^7 - 1)}{0.25 - 1}$

$S_7 = 65532$